$$\left(\frac{x_i\alpha - y_i\beta + z_i\gamma}{A}, \frac{x_i\alpha' + y_i\beta' + z_i\gamma'}{B}, \frac{x_i\alpha'' + y_i\beta'' + z_i\gamma''}{C}\right)$$

where

$$x_{1,3} = \lambda_{1,3}(\sqrt{g_+} \pm \sqrt{g_-}), \quad y_{1,3} = \lambda_{1,3}(\sqrt{g_+} \mp \sqrt{g_-})$$
$$z_{1,3} = (g_{12} \mp g_{23})\sqrt{g_+} + (g_{23} \pm g_{12})\sqrt{g_-}$$
$$g_\pm = {}^1/_2(g_{11} + g_{33}) \pm g_{13}, \quad x_2 = y_2 = 0, \quad z_2 = 1$$

Since $V = a(\lambda_1 + \lambda_3)$, $U = -a\lambda_1\lambda_3$, the characteristic functions are $\Phi_{1,3} = a\lambda_{1,3}^2 + h\lambda_{1,3} - f$, $\Phi_2 = U - f$.

The regions where motion is possible $M_{fh} = \{\Phi_1 \geqslant 0\} \cap \{\Phi_3 \leqslant 0\}$ are bounded by the surfaces $\{\lambda_{1,3} = \text{const}\}$ and are determined by the disposition of the roots of the quadratic trinomial $a\lambda^2 + h\lambda - f$ with respect to the intervals $[C, A]$ and $[-A, -C]$.

The critical points of the characteristic functions are defined by the equations

$$0 = d\Phi_{1,3} = (2a\lambda_{1,3} + h)\, d\lambda_{1,3}, \quad 0 = d\Phi_2 = dU$$

The conditions $dU = 0$, $d\lambda_{1,3} = 0$ provide the points $\mathbf{u}_{A,B,C} \parallel \mathbf{e}_{1,2,3}$, determined earlier, and the conditions $2a\lambda_{1,3} + h = 0$ define new sets of critical points that is, the surfaces $\{\lambda_{1,3} = \text{const}\}$. To the critical zero level of the characteristic function $\Phi_1|_{fh}$ there corresponds the constant quantity λ_1 equal to the multiple root of the trinomial $a\lambda^2 + h\lambda$ in the interval $[C, A]$. This trinomial is positive on $[-A, -C]$, i.e. everywhere we have $\Phi_3|_{fh} > 0$, and hence M_{fh} is empty. On the critical zero level $\{\lambda_3 = \text{const}\}$ of the characteristic function $\Phi_3|_{fh}$ we similarly have $\Phi_1|_{fh} > 0$, i.e. $M_{fh} = \{\lambda_3 = \text{const}\}$.

By the second theorem in Sect.2 this surface M_{fh} consists of the trajectories of the problem on which the velocity $\mathbf{v} = \omega\mathbf{w}_3$, $\omega = \pm\sqrt{2(h - V)}\,|\mathbf{w}_3|^{-1}$.

Using the integral G, we obtain the function of positional variables $G(\omega\mathbf{w}_3)$ that are functionally independent of λ_3. Since on these trajectories $G(\omega\mathbf{w}_3) = \text{const}$, $\lambda_3 = \text{const}$, they are closed curves and the motions on them are periodic.

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NON-LINEAR ANALYSIS OF THE STABILITY OF THE LIBRATION POINTS OF A TRIAXIAL ELLIPSOID*

I.I. KOSENKO

The stability of libration points of a triaxial homogeneous gravitating ellipsoid rotating around one of its principal central axes of inertia is studied. The plane motion of a passive point of unit mass is considered. In parameter space a region of stability is constructed and, also, resonance sets for all the resonances investigated. A systematic analysis of the stability of a libration point is carried out, using respective theorems for the equilibrium positions of Hamiltonian systems with two degrees of freedom.

A qualitative investigation of the geometric structure of the stability region was carried out in /1, 2/.

If the ellipsoid is a figure of revolution around the central polar axis of inertia /1/, the relative equilibrium positions are not isolated and fill a circle in the equatorial plane. If, however, the equatorial semiaxes are different, the ellipsoid may have up to four isolated positions of relative equilibrium. The conditions of existence of libration points external to the ellipsoid in this problem and, also, the canonical equations of motion in the

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neighbourhood of the equilibrium position investigation were obtained in /1/. A suitable space of the mechanical parameters of the problem was introduced and utilized in /1, 2/.

1. The stability region. The equations of motion of a passive point in the neighbourhood of relative equilibrium have the form

$$q_i^{\cdot} = H_{p_i}, \quad p_i^{\cdot} = -H_{q_i} \quad (i = 1,2) \tag{1.1}$$

in which the Hamiltonian is expanded in the power series

$$H(\mathbf{q}, \mathbf{p}) = H_2(\mathbf{q}, \mathbf{p}) + H_3(\mathbf{q}, \mathbf{p}) + H_4(\mathbf{q}, \mathbf{p}) + \ldots \tag{1.2}$$

The homogeneous forms $H_k(\mathbf{q}, \mathbf{p})$ $(k = 2, 3, 4)$ are represented here as follows:

$$\begin{aligned} &H_2(\mathbf{q}, \mathbf{p}) = h_{2000}q_1^2 + h_{1001}q_1p_2 + h_{0200}q_2^2 + h_{0110}q_2p_1 + \\ &\quad h_{0020}p_1^2 + h_{0002}p_2^2 \\ &H_3(\mathbf{q}, \mathbf{p}) = h_{3000}q_1^3 + h_{1200}q_1q_2^2, \; H_4(\mathbf{q}, \mathbf{p}) = h_{4000}q_1^4 + \\ &\quad h_{2200}q_1^2q_2^2 + h_{0400}q_2^4 \end{aligned} \tag{1.3}$$

The coefficients of the Taylor series, which will be subsequently required, are calculated in the form

$$\begin{aligned} &h_{2000} = -(\varphi_2 + \varphi_3)/(2\varphi_1), \quad h_{1001} = -1, \quad h_{0200} = \varphi_2/(2\varphi_1) \\ &h_{0110} = 1, \; h_{0020} = h_{0002} = {}^1/_2 \\ &h_{3000} = (1/\beta_2^2 + 1/\beta_3^2)/(3\varphi_1\beta_2\beta_3), \quad h_{1200} = 1/(\varphi_1\beta_2^2\beta_3) \\ &h_{4000} = -\xi_0[(1/\beta_2^2 + 1/\beta_3^2)^2 + 2(1/\beta_2^4 + 1/\beta_3^4)]/(12\varphi_1\beta_2\beta_3) \\ &h_{2200} = \xi_0(3/\beta_2^2 + 1/\beta_3^2)/(2\varphi_1\beta_2^3\beta_3), \quad h_{0400} = -\xi_0/(4\varphi_1\beta_2^5\beta_3) \\ &\beta_i = (\nu + \alpha_i)^{1/2} \quad (i = 2, 3) \end{aligned} \tag{1.4}$$

where the mechanical parameters of the problem are α_1, α_2, ν (the parameter space is three-dimensional). The remaining quantities $\alpha_3 = 1 - \alpha_1 - \alpha_2$, $\nu = \xi_0^2 - \alpha_1$ depend on them. Here α_1 is the square of the equatorial semiaxis of the ellipsoid on whose continuation lies the libration point investigated here /1/, α_2 is the square of the equatorial semiaxis, α_3 is the square of the polar semiaxis (a dependent parameter), and ξ_0 is the distance from the centre of the ellipsoid to the libration point. All the quantities are dimensionless.

Later the following elliptic parameters /1/ will be required:

$$\varphi_i = \int_{\nu}^{+\infty} \{(\alpha_i + u)^{-1}[(\alpha_1 + u)(\alpha_2 + u)(\alpha_3 + u)]^{-1/2}\}\, du \quad (i = 1, 2, 3) \tag{1.5}$$

The parameter space of the problem is a prime defined by the formula

$$\Pi^3 = \{(\alpha_1, \alpha_2, \nu) : \alpha_1 > 0, \; \alpha_2 > 0, \; \alpha_1 + \alpha_2 < 1, \; \nu > 0\} \tag{1.6}$$

The normal oscillation frequencies of plane motion are

$$\begin{aligned} &\omega_{1,2} = \{1 - h/2 \pm [(g + h/2)^2 - 2h]^{1/2}\}^{1/2}, \; 0 \leqslant \omega_2 \leqslant \omega_1 \\ &g = \varphi_2/\varphi_1, \; h = \varphi_3/\varphi_1 \end{aligned} \tag{1.7}$$

The qualitative results in /1, 2/ enabled us to construct subsequently the exact stability region and the resonance surfaces (Fig.1). Formulae (1.7) and (1.5) define the mapping Ω : $(\alpha_1, \alpha_2, \nu) \mapsto (\omega_1, \omega_2)$ of the parameter space Π^3 onto the particular space $\mathbf{R}_+^2 = \{(\omega_1, \omega_2)\colon \omega_1 \geqslant 0, \omega_2 \geqslant 0\}$. In $\mathbf{R}_+^2$ the stability conditions of the system, to a first approximation, have the form $0 < \omega_2 < \omega_1$, $\omega_1^2 + \omega_2^2 < 2$ (Fig.2) and determine the set $\Sigma \subset \mathbf{R}_+^2$. We denote by $\mathrm{St} = \Omega^{-1}(\Sigma)$ the stability region in Π^3. The length of the arc $\omega_1^2 + \omega_2^2 = 2$ correspond to the condition $h = 0$ or $\varphi_3 = 0$. From the mechanical meaning of the problem it is clear that always $\varphi_3 > 0$ (since the quantities $\alpha_i > 0$, $i = 1, 2, 3$). This inequality leads to the relation $\omega_1^2 + \omega_2^2 < 2$. The sets that correspond to one another in the mapping $\Omega : \Pi^3 \to \mathbf{R}_+^2$, are indicated in Figs.1 and 2 by the same numerals. The resonance surfaces *1 — 4* (Fig.1) of the first $(\omega_2 = 0)$, second $(\omega_1 = \omega_2)$, third $(\omega_1 = 2\omega_2)$, and fourth order $(\omega_1 = 3\omega_2)$ correspond to the resonance straight lines *1 — 4* (Fig.2). The preimage of double resonance $(\omega_1 = \omega_2 = 0)$ is the rectilinear interval 5 (Fig.1).

Let us determine the boundary of the stability region $\mathrm{St} \subset \Pi^3$ (in topology Π^3). It consists of the following: 1) semi-infinite section S_1 (set 1) of the plane $\alpha_1 = \alpha_2$, bounded by straight lines: 5; $\alpha_1 = \alpha_2 = {}^1/_2$, $\alpha_1 = \alpha_2$, $\nu = 0$, and $\alpha_1 = \alpha_2 = 0$ are surface of first-order resonances, 2) surfaces of second-order resonances S_2 (set 2); and 3) the interval of double resonance $\omega_1 = \omega_2 = 0$. In addition we shall consider surfaces S_3 and S_4 of the third-and fourth-order resonances (the sets *3* and *4* in Fig.1). Obviously $S_3 \subset \mathrm{St}$ and $S_4 \subset \mathrm{St}$. The stability of the resonance set $S_0 = St \setminus (S_3 \cup S_4)$ will be separately checked. The coordinates

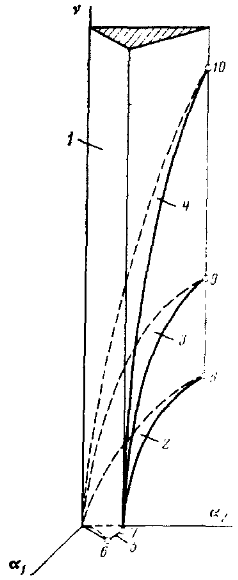

Fig.1

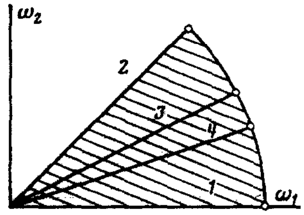

Fig.2

of the points $6-10$ (Fig.1) calculated in Π^3 are $(0.43\ldots; 0.43\ldots; 0)$, $(0.5; 0.5; 0.2408\ldots)$, $(0; 1; 6.2308\ldots)$, $(0; 1; 10, 2799\ldots)$ $(0; 1; 19, 0293\ldots)$.

2. The first-order resonance. The first-order resonance oocurs only at the boundary of the stability region to a first approximation and corresponds to the case of the second zero frequency $\omega_2 = 0$. Hence at the boundary S_1, including the interval 5 of double resonance, the equation $g = 1$ holds, from which $\varphi_1 = \varphi_2$. This in turn leads to the conclusion that for the first-order resonance we have an ellipsoid of revolution with equal equatorial semiaxes $\alpha_1 = \alpha_2$. The libration point, if it exists, is not isolated. The relative equilibria occupy the entire circle of the equatorial plane, and in the inertial frame of reference are solutions of the equation of motion of a satellite in a circular orbit.

Such solutions are generally Lyapunov unstable, if all phase variables are taken into account. This is due to the systematic shift along the equatorial orbit longitude. Because of this, the stability of circular orbits is usually investigated for some of the variables /3/. In the present paper the stability of the libration point means the stability of the position of relative equilibrium in a rotating system of coordinates or the stability of periodic motion over all phase variables.

For example, following /3/ it is possible to check that for any ellipsoid of revolution there is a shift along the circular orbit longitude. Hence we may conclude that in the case of a single zero frequency $\omega_2 = 0$ for first-order resonance (also for double resonance $\omega_1 = \omega_2 = 0$). the libration point in this problem is always unstable.

3. The second-order resonance. We shall carry out a non-linear analysis of the stability for $(\alpha_1, \alpha_2, \nu) \in S_2$ as in /4, 5/. The equations of plane motion in the neighbourhood of a libration point using local canonical variables may be represented in the form $\mathbf{z}^{\cdot} = IH_{\mathbf{z}}$, where $\mathbf{z}$ is a four-dimensional vector $\mathbf{z} = \mathrm{col}\,(q_1, q_2, p_1, p_2)$. and I is a simplectic fourth-order matrix $I^2 = -E$. The Hamiltonian function is represented as in (1.2) in the form of the series

$$H(\mathbf{z}) = H_2(\mathbf{z}) + H_3(\mathbf{z}) + \ldots$$

It can be shown that on surface S_2 of second-order resonance (the case of equal frequencies $\omega_1 = \omega_2 = \omega$) the minor M_{14} of the matrix $I(H_2)_{\mathbf{zz}} - i\omega E$ (the first row and the fourth column are deleted) is non-zero. Hence, when $(\alpha_1, \alpha_2, \nu) \in S_2$, the eigenvalue $i\omega$ has a non-prime elementary divisor. According to /4/ by a linear canonical transformation $\mathbf{z} = N\mathbf{Z}$ ($\mathbf{Z} = \mathrm{col}\,(Q_1, Q_2, P_1, P_2)$ the Hamiltonian function can be reduced to form such that the homogeneous form of the lowest order of H_2 can be represented as follows: $H_2(\mathbf{z}) = F_2(\mathbf{Z}) = (P_1^2 + P_2^2)/2 + \omega(Q_1P_2 - Q_2P_1)$

where the frequency is defined by the equation $\omega = (1 - h/2)^{1/2}$. The matrix of simplectic transformation has the form

$$N = (a_{ij})(i, j = 1, \ldots, 4),\ a_{11} =$$
$$-1/(1 + a)^{1/2},\quad a_{22} = 1/(1 - a)^{1/2},\ a_{41} = a/(1 + a)^{1/2}$$
$$a_{32} = -a/(1 - a)^{1/2},\quad a_{33} = -(2 - a^2)/\{2[(1 - a^2)(1 - a)]^{1/2}\}$$
$$a_{44} = (2 - a^2)/\{[(1 - a^2)(1 + a)]^{1/2}\},\ a_{23} = a/\{2[1 - a^2)(1 - a)]^{1/2}\}$$
$$a_{14} = -a/\{2[(1 - a^2)(1 + a)]^{1/2}\},\ a = (h/2)^{1/2}$$

The remaining elements of the matrix N are zero. The transformed Hamiltonian is defined by the equation

$$F(\mathbf{Z}) = F_2(\mathbf{Z}) + F_3(\mathbf{Z}) + \ldots, \quad F_k(\mathbf{Z}) = \sum_{i_1+i_2+j_1+j_2=k} f_{i_1 i_2 j_1 j_2} Q_1^{i_1} Q_2^{i_2} P_1^{j_1} P_2^{j_2} \tag{3.1}$$

where $F_k(\mathbf{Z})$ are homogeneous forms of power k. By a non-linear normalizing transformation $\mathbf{Z} \mapsto \zeta$ the Hamiltonian function can be reduced to the form

$$\Lambda(\zeta) = \frac{1}{2}(\eta_1^2 + \eta_2^2) + \omega(\xi_1\eta_2 - \xi_2\eta_1) + A(\xi_1^2 + \xi_2^2) + \sum_{k=5}^{\infty} \Lambda_k(\zeta)$$

$$\zeta = \mathrm{col}(\xi_1, \xi_2, \eta_1, \eta_2)$$

The following theorem holds /5/: if $A > 0$, the equilibrium position is Lyapunov stable, and when $A < 0$, it is Lyapunov unstable.

Formulae obtained in /5/ must be used to calculate the coefficient A. A computer check showed that for any $(\alpha_1, \alpha_2, \nu) \in S_2$ the quantity $A > 0$, and hence the libration point is Lyapunov stable.

4. **Third-order resonance**. The set in parameter space where resonance relations for third-order resonance are satisfied, is non-empty, and as already indicated in Sect.1, is defined by the equation $S_3 = \Omega^{-1}\{(\omega_1, \omega_2) : \omega_1 = 2\omega_2\}$ (S_3 is a surface in space Π^3).

Like any layer in St defined by the equation

$$S_{k+1} = \Omega^{-1}\{(\omega_1, \omega_2) : \omega_1 = k\omega_2, k > 1, k \in \mathbf{R}\} \tag{4.1}$$

the surface S_3 is projected one-to-one on to the plane of parameter (α_1, α_2) (in Fig.3 this projection is shaded). The interpretation of results of stability investigations in cases of resonance is clear and convenient, since it is possible to talk about this prooperty in terms of the values of the two equatorial semiaxes of the ellipsoid. We present in the same way the diagram of the stability and instability regions of the fourth-order resonance. It is thus possible to judge the stability of the libration point by the form of the ellipsoid.

Later we shall have to study the problem at inner points of the region St. At the boundary of St the investigation has been completed in Sects.2 and 3, since $\mathrm{Fr}\,\mathrm{St} = S_1 \cup S_2 \cup I_{12}$, where I_{12} is the rectilinear interval 5 (Fig.1) of double resonance $\omega_1 = \omega_2 = 0$, determined by relations $\alpha_1 = \alpha_2$, $(1-c)\nu + 1 = (2+c)\alpha_1$ where c is the solution of the equation

$$[(1-c)/c]^{1/2} + [c(1-c)]^{1/2} = \mathrm{arcctg}\,[c/(1-c)]^{1/2}$$

Region St does not contain resonances of the first and second order, and the roots of the characteristic equation of the first approximation are simple and purely imaginary. Hence a linear canonical transformation of system (1.1) exists of the form $\mathbf{z} = N\mathbf{Z}$ that reduces the quadratic form of expansion (1.2) of the Hamiltonian to the form

$$H_2(\mathbf{z}) = H_2(N\mathbf{Z}) = F_2(\mathbf{Z}) = \omega_1(Q_1^2 + P_1^2)/2 - \omega_2(Q_2^2 + P_2^2)/2 \tag{4.2}$$

(see, e.g., /6/). It can be shown that the simplectic matrix N has the form $N = (a_{ij})(i, j = 1, \ldots, 4)$, where only the elements

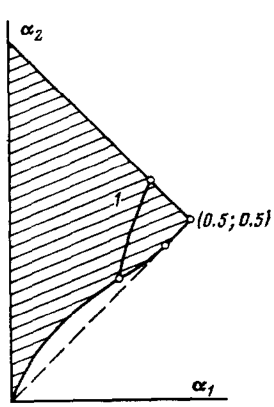

Fig.3

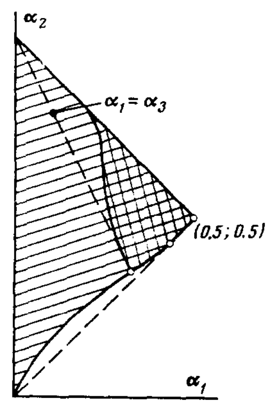

Fig.4

$$a_{13} = (1 - g + \omega_1^2)/(-\omega_1 B_1)^{1/2}, \quad a_{14} = (1 - g + \omega_2^2)/(\omega_2 B_2)^{1/2}$$
$$a_{21} = -2(-\omega_1/B_1)^{1/2}, \quad a_{22} = 2(\omega_2/B_2)^{1/2}$$
$$a_{31} = (1 + g - \omega_1^2)(-\omega_1/B_1)^{1/2}, \quad a_{32} = -(1 + g - \omega_2^2) \times (\omega_2/B_2)^{1/2}$$
$$a_{43} = (1 - g - \omega_1^2)/(-\omega_1 B_1)^{1/2}, \quad a_{44} = (1 - g - \omega_2^2)/(\omega_2 B_2)^{1/2}$$
$$B_i = 3 - 2g - g^2 + 2(g-1)\omega_i^2 - \omega_i^4 \quad (i = 1, 2)$$

are non-zero. The transformed Hamiltonian function is defined by (3.1). We further reduce the cubic terms in (3.1) to the normal form.* (*See Sect.3, Markeyev A.P. and Sokol'skii A.G. Some Computational Algorithms for Normalizing Hamiltonian Systems. Moscow. Preprint No.31, Inst. Prikl. Matem. AN SSSR, 1976.)

We will transfer complex canonical variables using the formulae

$$Q_1 = -(\xi_1 + i\eta_1)/\sqrt{2},\ P_1 = -(i\xi_1 + \eta_1)/\sqrt{2}$$
$$Q_2 = -(-\xi_2 + i\eta_2)/\sqrt{2},\ P_2 = -(i\xi_2 - \eta_2)/\sqrt{2}$$

Let us obtain the new representation of the Hamiltonian function (its quadratic part) $F_2(\mathbf{Z}) = \Lambda_2(\xi, \eta) = i\omega_1\xi_1\eta_1 + i\omega_2\xi_2\eta_2$. To normalize the third-order terms it is necessary to solve the operator equation $DT_3 = G_3 - K_3$, where the differential operator is determined using the Poisson bracket

$$Df = \{f, \Lambda_2\} = \sum_{j=1}^{2}\left(\frac{\partial\Lambda_2}{\partial\eta_j}\cdot\frac{\partial f}{\partial\xi_j} - \frac{\partial\Lambda_2}{\partial\xi_j}\cdot\frac{\partial f}{\partial\eta_j}\right), \qquad G_3(\xi, \eta) = F_3(\mathbf{Z})$$

where T_3 is the third-order form in the expansion of the generating function of Lie transformulation in the Hori method, and K_3 is the normal form of the third-order Hamiltonian. In K_3 only the resonance terms $K_3(\xi, \eta) = K_{1002}\xi_1\eta_2^2 + K_{0210}\xi_2^2\eta_1$ remain, and $K_{1002} = i(f_{0012} - f_{0210} - f_{1101})$, $K_{0210} = iK_{1002}$.

We obtain the following result /6/. If the Hamiltonian function is such that $|K_{1002}| = |K_{0210}| \neq 0$, the libration point is unstable. When $|K_{1002}| = 0$ and the conditions of the Arnold-Moser theorem are satisfied (see Sect.6) (the third-order resonance does not in this case impede further normalization), we have Lyapunov stability.

A computer check showed that everywhere on S_3 the condition of instability is satisfied, except along the curve whose projection *1* is shown in Fig.3. Along that curve the resonance terms which impede the stability do not exist in the expansion of the Hamiltonian, and the fourth-order resonance is not satisfied. Hence the normalization required by the Arnold-Moser theorem /7/ is possible. A check proved that according to that theorem the libration curve *1* is stable.

5. Fourth-order resonance. Let us investigate the stability of the libration point of the parameter set S_4 (see Sect.1). As in Sect.4, we will interpret the results not in Π^3 but in the (α_1, α_2) plane where the surface S_4 is projected one-to-one (the shaded region in Fig.4). When the fourth-order resonance relation $\omega_1 = 3\omega_2$ is satisfied (in this problem of plane motion there is not other resonance of this order) to analyse the stability we have to reduce H to the normal form up to the fourth order inclusive.

It is assumed that the Hamiltonian function is reduced to the form (3.1), where $F_2(\mathbf{Z})$ has the form (4.2). As noted in Sect.4, this can be done using a suitable linear canonical transformation.

Let us change to canonical polar variables, using the formulae

$$Q_k = (2r_k)^{1/2}\sin\varphi_k,\ P_k = (2r_k)^{1/2}\cos\varphi_k\ (k = 1,\ 2) \tag{5.1}$$

In the new variables the Hamiltonian has the form

$$K(r_1, r_2, \varphi_1, \varphi_2) = \sum_{k=2}^{\infty} K_k(r_1, r_2, \varphi_1, \varphi_2), \qquad K_2 = \omega_1 r_1 - \omega_2 r_2$$
$$K_m = (1/2)^{(m-2)/2}\sum_{k_1+k_2=m} U_{k_1k_2}(\varphi_1, \varphi_2)(r_1^{k_1}r_2^{k_2})^{1/2} \quad (m = 3, \ldots)$$

where $U_{k_1k_2}(\varphi_1, \varphi_2)$ are trigonometric polynomials calculated in terms of respective coefficients of the forms F_m (see below). Since there are no third-order resonances, the Hamiltonian has the following normal form, obtained after the non-linear canonical exchange of variables $(r_1, r_2, \varphi_1, \varphi_2) \mapsto (\rho_1, \rho_2, \psi_1, \psi_2)$: $H = G_2 + G_4 + R + \ldots$. Terms in the expansion of H of order higher than the fourth are omitted, and

$$G_2 = \omega_1\rho_1 - \omega_2\rho_2,\ G_4 = C_{20}\rho_1^2 + C_{11}\rho_1\rho_2 + C_{02}\rho_2^2 \tag{5.2}$$

The resonance term for commensurability $\omega_1 = 3\omega_2$ has the form

$$R = [A_{13}\sin(\psi_1 + 3\psi_2) + B_{13}\cos(\psi_1 + 3\psi_2)](\rho_1\rho_2^3)^{1/2}$$

Let us determine the quantities $a = C_{20} + 3C_{11} + 9C_{02}$, $b = 3[3(A_{13}^2 + B_{13}^2)]^{1/2}$. The following Markeyev theorem holds /6/. If the Hamiltonian of perturbed motion is such that $|a| < b$, the equilibrium position is Lyapunov unstable; if however $|a| > b$, it is Lyapunov stable.

To verify these conditions it is evidently necessary to determine the coefficients of the functions G_4 and R. The coefficients of the resonance term are calculated using the following series of formulae:

$$
\begin{aligned}
&A_{13} = a_4{}^{13}/2 - \alpha_4{}^{13}/8, \quad B_{13} = b_4{}^{13}/2 - \beta_4{}^{13}/8 \\
&\alpha_4{}^{13} = 2d_{01}(\gamma_{1,03}^{3,12} + \gamma_{2,03}^{1,12}) + 2d_{12}(3\gamma_{3,12}^{1,03} + \gamma_{3,12}^{1,21}) + (d_{21} - d_{-1,2})\,\gamma_{2,12}^{3,21} \cdot 2 \\
&\beta_4{}^{13} = 2d_{01}(\delta_{1,03}^{3,12} + \delta_{2,03}^{1,12}) + 2d_{12}(3\delta_{3,12}^{1,03} + \delta_{3,12}^{1,21}) + (d_{21} - d_{-1,2})\,\delta_{2,12}^{3,21} \cdot 2 \\
&\gamma_{i,\,k_1k_2}^{j,\,n_1n_2} = a_i^{k_1k_2} b_j^{n_1n_2} + b_i^{k_1k_2} a_j^{n_1n_2}, \quad \delta_{i,\,k_1k_2}^{j,\,n_1n_2} = -a_i^{k_1k_2} a_j^{n_1n_2} + b_i^{k_1k_2} b_j^{n_1n_2}
\end{aligned}
\tag{5.3}
$$

Let us determine the quantities $d_{n_1n_2} = (n_1\omega_1 - n_2\omega_2)^{-1}$. The quantities $a_i^{k_1k_2}$ and $b_j^{n_1n_2}$ are coefficients of trigonometric polynomials $U_{k_1k_2}(\varphi_1, \varphi_2)$ and $U_{n_1n_2}(\varphi_1, \varphi_2)$ respectively. The formulae used to obtain the coefficients of the form G_4 are also applicable for normalizing in the non-resonance case up to the fourth order inclusive. Hence they will be used to check that the Arnold-Moser conditions in Sect.6 are satisfied. These formulae are

$$
\begin{aligned}
&C_{k_1k_2} = c_{2k_1,\,2k_2}/2 - \gamma_{2k_1,\,2k_2}/8 \\
&\gamma_{40} = 3d_{10}(\beta_{1,30} + \beta_{2,30}) + d_{01}\beta_{1,21} - d_{2,-1}\beta_{2,21} + d_{21}\beta_{3,21} \\
&\gamma_{22} = 4d_{10}\beta_{1,30}^{1,12} + 4d_{01}\beta_{1,03}^{1,21} + 4d_{2,-1}\,\beta_{2,21} + 4d_{21}\beta_{3,21} + \\
&\qquad 4d_{-1,2}\beta_{2,12} + 4d_{12}\beta_{3,12} \\
&\gamma_{04} = 3d_{01}(\beta_{1,03} + \beta_{2,03}) + d_{10}\beta_{1,12} - d_{-1,2}\beta_{2,12} + d_{12}\beta_{3,12} \\
&\beta_{i,\,k_1k_2} = (a_i^{k_1k_2})^2 + (b_i^{k_1k_2})^2, \qquad \beta_{i,\,k_1k_2}^{j,\,n_1n_2} = a_i^{k_1k_2} a_j^{n_1n_2} + b_i^{k_1k_2} b_j^{n_1n_2}
\end{aligned}
\tag{5.4}
$$

The quantities $a_i{}^{30}$, $b_i{}^{30}$, $c_{2k_1,\,2k_2}$ are the coefficients of trigonometric polynomials U_{30}, U_{40}, U_{22}, U_{04}.

In all equations (5.3) and (5.4) the quantities $a_i^{k_1k_2}$, $b_j^{n_1n_2}$, $c_{2k_1,\,2k_2}$ are explicitly expressed in terms of the coefficients of the Hamiltonian power expansion by the formulae

$$
\begin{aligned}
&a_4{}^{13} = f_{1003} + f_{0112} - f_{1201} - f_{0310}, \quad b_4{}^{13} = f_{1300} + f_{0013} - \\
&\qquad f_{1102} - f_{0211} \\
&a_1{}^{03} = 3f_{0300} + f_{0102}, \quad b_3{}^{12} = -f_{0210} + f_{0012} - f_{1101} \\
&b_1{}^{03} = f_{0201} + 3f_{0003}, \quad a_3{}^{12} = -f_{1200} + f_{1002} + f_{0111} \\
&a_2{}^{03} = -f_{0300} + f_{0102}, \quad b_1{}^{12} = 2(f_{0210} + f_{0012}) \\
&b_2{}^{03} = -f_{0201} + f_{0003}, \quad a_1{}^{12} = 2(f_{1200} + f_{1002}) \\
&b_1{}^{21} = 2(f_{2001} + f_{0021}), \quad a_1{}^{21} = 2(f_{2100} + f_{0120}) \\
&a_2{}^{12} = f_{1200} - f_{1002} + f_{0111}, \quad b_3{}^{21} = -f_{2001} + f_{0021} - f_{1110} \\
&b_2{}^{12} = -f_{0210} + f_{0012} + f_{1101}, \quad a_3{}^{21} = -f_{2100} + f_{0120} + f_{1011} \\
&c_{40} = 3f_{4000} + f_{2020} + 3f_{0040} \\
&c_{22} = 2(f_{2200} + f_{0220} + f_{2002} + f_{0022}) \\
&c_{04} = 3f_{0400} + f_{0202} + 3f_{0004} \\
&a_1{}^{30} = 3f_{3000} + f_{1020}, \quad b_1{}^{30} = f_{2010} + 3f_{0030} \\
&a_2{}^{30} = -f_{3000} + f_{1020}, \quad b_2{}^{30} = -f_{2010} + f_{0030} \\
&a_2{}^{21} = f_{2100} - f_{0120} + f_{1011}, \quad b_2{}^{21} = -f_{2001} + f_{0021} + f_{1110}
\end{aligned}
$$

The results of a computer check that the conditions of Markeyev's theorem are satisfied are shown in Fig.4. The instability region (more exactly, its one-to-one projection on the plane (α_1, α_2)) is indicated by cross-hatching. To investigate the stability along the curve separating the instability region from that of stability on the surface S_4 it is necessary, in the expansion of the Hamiltonian, terms of power higher than the fourth.

The above analysis shows that when the fourth-order resonance is satisfied in this problem, loss of stability occurs only when $\alpha_3 < \alpha_1$, i.e. when the ellipsoid is oblate. With the compression increases the libration point remains unstable.

6. The non-resonance case.

The set S_0 corresponds to this case (see Sect.1). Note that as $\nu \to \infty$ this set is not bounded in Π^3. Since it is three-dimensional, it is convenient to, carry out the analysis on sections of S_0 by planes $\nu = \text{const}$ $(\nu > 0)$ and observe the variation of the stability conditions as ν changes. The basis of this analysis is the Arnold-Moser theorem. If there are no resonances up to and including the fourth order, it is *possible by a suitable canonical change of variables to reduce the Hamiltonian function to the form*

$$G(\rho_1, \rho_2, \psi_1, \psi_2) = G_2(\rho_1, \rho_2) + G_4(\rho_1, \rho_2) + \sum_{k=5}^{\infty} G_k(\rho_1, \rho_2, \psi_1, \psi_2)$$

where ρ_i, ψ_i $(i = 1, 2)$ are canonical polar coordinates and the homogeneous forms G_2 and G_4 are determined by (5.2). Then, by the Arnold-Moser theorem, the equilibrium position of the Hamiltonian system (the libration point) is Lyapunov stable, if

$$V(\alpha_1, \alpha_2, \nu) = C_{20}\omega_2{}^2 + C_{11}\omega_1\omega_2 + C_{02}\omega_1{}^2 \neq 0$$

A check showed that, when there are no resonances (in region S_0) the stability of the libration point in plane motion occurs at all points S_0, except at points lying on the surface Γ. The set Γ is bounded and contained between the sets S_3 and S_4. Calculations showed that on Γ the function $V = 0$, and that the quantity V changes its sign on passing through Γ. To investigate the stability on the surface $\Gamma = \{(\alpha_1, \alpha_2, \nu): V(\alpha_1, \alpha_2, \nu) = 0\}$ it is necessary to use in the power expansion of the Hamiltonian, terms of order higher than the fourth. On the set $\bigcup S_{k+1}$ $(1 < k < 2)$ the function $V > 0$. In passing through the surface S_3 the third-order resonance function V changes its sign, and has a discontinuity of the second kind on S_3. In the subregion S_0 bounded by the surfaces S_3 and Γ, the quantity V is negative. Then, on passing through Γ, it again changes its sign and remains positive in the remaining part of S_0.

An idea of the disposition of Γ in S_0 can be obtained by considering several sections $S_0^{(\nu)} = \{(\alpha_1, \alpha_2, \nu'): \nu' = \nu\}$. Four typical cases are possible, corresponding to four intervals

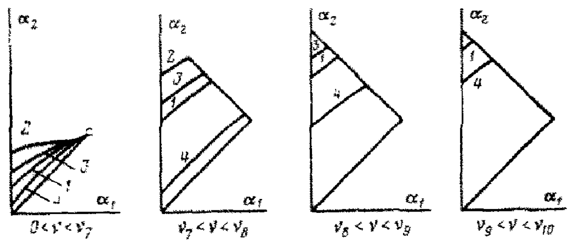

Fig.5

ν: $0 < \nu < \nu_7$, $\nu_i < \nu < \nu_{i+1}$ $(i = 7, 8, 9)$, where ν_i $(i = 7, 8, 9, 10)$ are remoteness parameters of points of Π^3 numbered 7–10 (Fig.1). These cases are represented in Fig.5, where the numerals 1, 2, 3, 4 denote the curves of intersection of $S_0^{(\nu)}$ with the surfaces Γ, S_2, S_3, S_4, respectively. Note that when $\nu > \nu_{10}$, and there are no resonance points in sections $S_0^{(\nu)}$ the section $S_0^{(\nu)} \cap \Gamma$ is also empty. Thus in plane motion a reasonably remote libration point is always stable.

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